MATH 20D Spring 2023 Lecture 25.

Eigenvalues and Constant Coefficient Homogeneous Systems.

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- Office hours during finals week next Tuesday from 9am-12pm in HSS 4085.

Outline

Contents

- Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2-by-2 matrix with entries $a, b, c, d \in \mathbb{R}$.

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so that multiplication by A gives a function $\mathbf{v} \mapsto A\mathbf{v}$ from $\mathbb{C}^2$ to $\mathbb{C}^2$.

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Example

The vector $\mathbf{v} = \mathrm{col}(1, 0, 1)$ is an eigenvector of $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ with eigenvalue $\lambda = 3$.

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(c) *The matrix equation $A\mathbf{x} = \mathbf{0}$ does not admit a non-zero solution.*

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- Write $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for the 2-by-2 identity matrix.
- The equation $A\mathbf{v} = \lambda\mathbf{v}$ may be rearranged to the form

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Definition

An ***eigenvalue of*** A *is a scalar* $\lambda \in \mathbb{C}$ *such that that equation*

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

admits a non-zero solution $\mathbf{v} \neq 0$.

- In light of the theorem on the previous slide, we see that the eigenvalues of A are exactly the values λ for which $\det(A - \lambda I) = 0$.

Example

Determine the eigenvalues of the following matrices

$$\textbf{(a)} \quad \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \qquad \textbf{(b)} \quad \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

- Given an eigenvalue λ of a matrix A, we can determine the corresponding eigenvectors by solving the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$.
- We can always "pull scalars through matrix multiplication" so if $(A - \lambda I)\mathbf{v} = \mathbf{0}$ then for example

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- Hence if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ then $2\mathbf{v}$ is also an eigenvector of A with eigenvalue λ. We could even replace 2 with any **non-zero scalar** and the same statement would hold.

- If $\mathbf{v}$ is an eigenvector for A with eigenvalue λ and s is a **non-zero scalar** then $s\mathbf{v}$ is another eigenvector for A with eigenvalue λ.
- Since they only differ by multiplication by a constant, we'd like to consider $\mathbf{v}$ and $s \cdot \mathbf{v}$ as more or less the "same" eigenvector of A.

Definition

Suppose $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^2$. We say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent if there does not exists a scalar $s \in \mathbb{C}$ such that

$$\mathbf{v}_1 = s \cdot \mathbf{v}_2.$$

Example

For each of the matrix below, find two linearly independent eigenvectors.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Contents

- Let A be a 2-by-2 matrix with constant entries and consider a system of differential equations in normal form

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t) = \mathrm{col}(x_1(t), x_2(t))$.

- As was the case for scalar equations, we are interested in constructing a **general solution** to (1) from which we will derive solutions to IVP's.

Theorem

Suppose A has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Let r_1 and r_2 denote the eigenvalues of $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

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$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$$

where $\mathbf{x}_1(t) = e^{r_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{r_2 t}\mathbf{v}_2$.

Example

Write down a general solution to the equation

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t)$$